Section 3.3

Maxima & minima for functions of n-Variables

Definitions

- If $f: U \subset \mathbb{R}^n \to \mathbb{R}$ then $x_0 \in U$ is called a local (maximum) minimum if there is a neighborhood of x_0 where $f(x) \geq f(x_0)$ ($f(x) \leq f(x_0)$)

- A local min or max is called a local extremum

- A critical point is a point x_0 where either f is not differentiable at x_0 or $(Df)(x_0) = 0$. A critical pt that is not an extremum is called a saddle point.

Theorem: If $f: U \subset \mathbb{R}^n \to \mathbb{R}$ (U open set) is differentiable & $x_0 \in U$ is a local extremum, then $(Df)(x_0) = 0$

(i.e., x_0 is a critical point of f)

Proof: Suppose x_0 is a local maximum (or min)

then for any direction $\vec{h}$, $g(t) = f(\vec{x}_0 + t\vec{h})$

has a local max (or min) at $t_0 = 0$, $\Rightarrow g'(0) = 0$

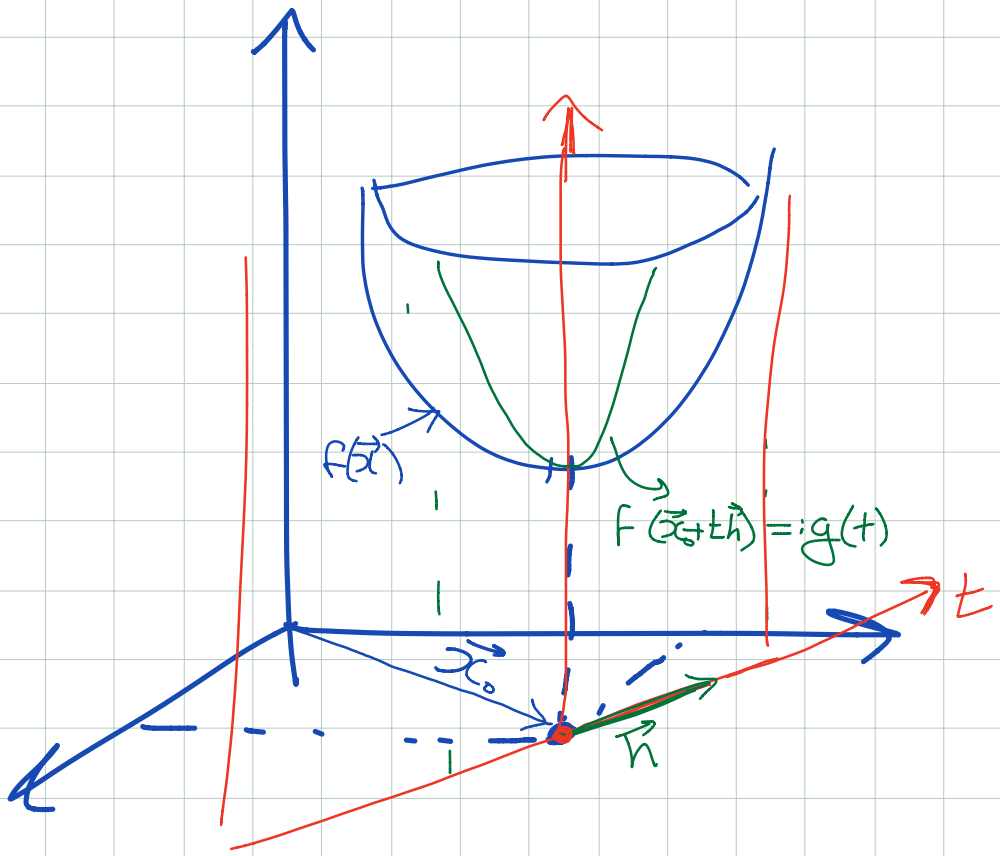
but $g'(t) = (Df)\big|_{\vec{x}_0 + t\vec{h}} \vec{h}$

$\Rightarrow g'(0) = (Df)(\vec{x}_0) \cdot \vec{h} = 0$

but this is true in every direction

$\Rightarrow (Df)(x_0) = 0$

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Remark: This just means that $(\nabla f)|_{\vec{x}_0} = 0$ or equivalently, that all the partial derivatives are zero.

Example: Find the critical points of the function $f(x,y) = x^2 + 4y^2$

Soln: The function is differentiable so we need to find the points where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

but $\frac{\partial f}{\partial x} = 2x$ & $\frac{\partial f}{\partial y} = 8y$

so the point $(0,0)$ is the only critical pt.

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Question

OK, so we now know how to find the critical points, but how to determine if they are a max, min, or saddle point?

Answer

For general functions $f: \mathbb{R}^n \to \mathbb{R}$ this can be fairly involved (see book)

However, we will deal with functions of two variables

Second derivative test for functions of two variables

Let $f(x,y)$ be a C^2 function on an open set $U \subset \mathbb{R}^2$ (so $f: U \subset \mathbb{R}^2 \to \mathbb{R}$)

Suppose that

(i) $\frac{\partial f}{\partial x}(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0) = 0$

(ii) $\frac{\partial^2 f}{\partial x^2}(x_0,y_0) \geq 0$ $(\leq)$

(iii) $D = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$ at (x_0,y_0)

D ↑ discriminant

then (x_0,y_0) is a local minimum (maximum)

If $D < 0$ at (x_0,y_0) then (x_0,y_0) is a saddle point.

If $D = 0$, the test fails

Examples

- Find the critical points of the function $f(x,y) = x^2 + y^4$ and classify them (using the 2nd der. test)

- Soln: $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 4y^3$

setting $\frac{\partial f}{\partial x} = 0 \Rightarrow x_0 = 0$

$\frac{\partial f}{\partial y} = 0 \Rightarrow y_0 = 0$

So our only critical point is $(0,0)$

Now, we must calculate

$$D = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

so we need $\frac{\partial^2 f}{\partial x^2} = 2$, $\frac{\partial^2 f}{\partial y^2} = 12y^2$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0 \Rightarrow D(0,0) = 24y^2\Big|_{(0,0)} = 0$$

so the test Fails.

On the other hand, graphing

$F(x,y)$ shows that $(0,0)$ is a min.

Example: Same question for.

$$F(x,y) = \log(x^2+y^2+1)$$

Soln:
$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2+1}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2+1}$$

$\Rightarrow$ the critical point is $(0,0)$

Now, we have to calculate D:

$$\frac{\partial^2 f}{\partial x^2}\Big|_{(0,0)} = \frac{2(x^2+y^2+1) - 2x(2x)}{(x^2+y^2+1)^2}\Big|_{(0,0)} = 2$$

Similarly $\left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} = 2$

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = \left.\frac{-4xy}{(x^2+y^2+1)^2}\right|_{(0,0)} = 0$$

$$\Rightarrow D(0,0) = (2)(2) - 0^2 = 4 > 0$$

$\Rightarrow (0,0)$ is a max or min

Now, we check $\partial^2 f / \partial x^2 (0,0) > 0$

$\Rightarrow (0,0)$ is a local minimum.

Global Maxima & Minima

Suppose $f: A \subset \mathbb{R}^2 \to \mathbb{R}$, A point $x_0 \in A$ is an absolute max (or min) if $f(x_0) \geq f(x)$ (or $\leq f(x)$) for all $x \in A$

Theorem: Let $D \subset \mathbb{R}^n$ be closed (i.e. contains all its boundary points) & bounded (i.e. $\|\vec{x}\| \leq M, \forall\, x \in D$) and let $f: D \to \mathbb{R}$ be continuous

then f achieves its absolute max and its absolute min on some points in D.

Strategy for finding Absolute maxima and minima on a region with a boundary

(i) Locate all the critical points in the region.

(ii) Find all the critical points of f viewed as a function on the boundary

(iii) Evaluate f at all the critical pts and select the largest & smallest

Example: Find the max and min values of

$$f(x,y) = x^2+y^2 - x - y + 1$$

in the disk D defined by $x^2+y^2 \leq 1$

Soln:

Step 1: Find the critical points in the interior of D.

$$\frac{\partial f}{\partial x} = 2x - 1 \quad , \quad \frac{\partial f}{\partial y} = 2y - 1$$

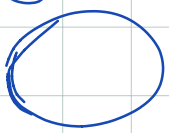
$$\frac{\partial f}{\partial x} = 0 \;\&\; \frac{\partial f}{\partial y} = 0 \Rightarrow x = \frac{1}{2}, \; y = \frac{1}{2}$$

So $(\frac{1}{2}, \frac{1}{2})$ is the only critical pt. in the interior of the disk

(note that $(\frac{1}{2})^2 + (\frac{1}{2})^2 < 1$ so $(\frac{1}{2}, \frac{1}{2}) \in D$)

Step 2: Must find the critical pts on the boundary of D.

The boundary of D is the circle

$x^2+y^2=1$ 

which can be parametrized as the set of points

$$(\cos\theta, \sin\theta), \quad \theta \in [0, 2\pi]$$

So $f(x,y) = f(\cos\theta, \sin\theta)$

$$= \cos^2\theta + \sin^2\theta - \cos\theta - \sin\theta - 1$$

$$= -\cos\theta - \sin\theta$$

To find the critical points on the circle we need:

$$\frac{df}{d\theta} = +\sin\theta - \cos\theta$$

and set $\frac{df}{d\theta} = 0$ to get $\sin\theta = \cos\theta$

$$\Rightarrow \theta = \frac{\pi}{4} \text{ or } \theta = \frac{5\pi}{4}$$

that is $(x,y) = (\cos\frac{\pi}{4}, \sin\frac{\pi}{4}) = (\sqrt{2}/2, \frac{\sqrt{2}}{\sqrt{2}})$

& $(-\sqrt{2}/2, -\sqrt{2}/2)$ are the cp's.

Step 3: $f(\frac{1}{2},\frac{1}{2}) = (\frac{1}{2})^2 + (\frac{1}{2})^2 - \frac{1}{2} - \frac{1}{2} + 1 =$

$= \frac{1}{2}$

Similarly $f(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}) = \frac{1}{2} + \frac{1}{2} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + 1$

$= 2 - \sqrt{2}$

& $f(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}) = 2 + \sqrt{2}$

$\Rightarrow$ $\frac{1}{2}$ is the minimum

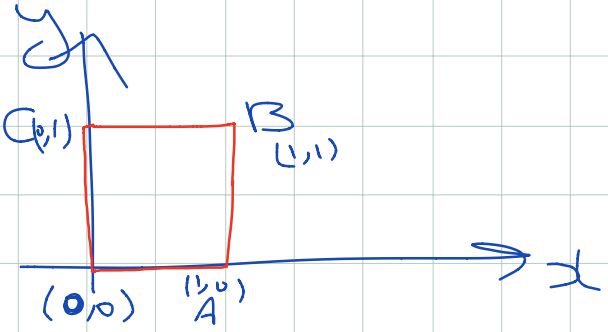
& $2+\sqrt{2}$ is the maximum.

Example:

Determine the global extreme values of the function $f(x,y) = x^4 - y$

where $0 \le x \le 1$, $0 \le y \le 1$

Solution



Step 1: Find the critical pts.

$$\frac{\partial f}{\partial x} = 4x^3 \quad , \quad \frac{\partial f}{\partial y} = -1$$

Since $\frac{\partial f}{\partial y} \neq 0$, there are no critical pts.

Step 2: Check the boundary.

We have to check each edge of the square separately.

(i) Segment OA: Here $y=0$, $0 \le x \le 1$

$\Rightarrow f(x,0) = x^4$

$\Rightarrow$ the max is $f(1,0) = 1$
the min is $f(0,0) = 0$

(ii) Segment AB: Here $x=1$, $0 \le y \le 1$

$\Rightarrow f(1,y) = 1 - y$

$\Rightarrow$ the max is $f(1,0) = 1$
the min is $f(1,1) = 0$

(iii) Segment BC : Here $y = 1$, $0 \le x \le 1$

$\Rightarrow f(x,1) = x^4 - 1$

$\Rightarrow$ the max is $F(1,1) = 0$
the min is $f(0,1) = -1$

(iv) Segment CO : Here $x = 0$, $0 \le y \le 1$

$\Rightarrow f(0,y) = 0 - y$

$\Rightarrow$ the max is $F(0,0) = 0$
the min is $f(0,1) = -1$

So, the smallest value is $f_{min} = -1$
and it is the global min on the square.

The largest value is $f_{max} = 1$
and it is the global max on the square.